ARENS REGULARITY OF CERTAIN WEIGHTED SEMIGROUP ALGEBRA AND COUNTABILITY

B. KHODSIANI[1], A. REJALI[2] AND H.R. EBRAHIMI VISHKI[3]

Abstract. It is known that every countable semigroup admits a weight ω for which the semigroup algebra $\ell_1(S,\omega)$ is Arens regular and no uncountable group admits such a weight; see [4]. In this paper, among other things, we show that for a large class of semigroups, the Arens regularity of the weighted semigroup algebra $\ell_1(S,\omega)$ implies the countability of S.

1. Introduction and Preliminaries

Arens [2] introduced two multiplications on the second dual $\mathfrak{A}^{**}$ of a Banach algebra $\mathfrak{A}$ turning it into Banach algebra. If these multiplications are coincide then $\mathfrak{A}$ is said to be Arens regular. The Arens regularity of the semigroup algebra $\ell_1(S)$ has been investigated in [7]. The Arens regularity of the weighted semigroup algebra $\ell_1(S,\omega)$ has been studied in [4] and [3]. In [3] Baker and Rejali obtained some nice criterions for Arens regularity of $\ell_1(S,\omega)$. Recent developments on the Arens regularity of $\ell_1(S,\omega)$ can be found in [5]. For the algebraic theory of semigroups our general reference is [6].

In this paper we first show that the Arens regularity of a weighted semigroup algebra is stable under certain homomorphisms of semigroups (Lemma 2.2). Then we study those conditions under which the Arens regularity of $\ell_1(S,\omega)$ necessities the countability of S. The most famous example for such a semigroup is actually a group, as Craw and Young have proved in their nice paper [4]. As the main aim of the paper we shall show that for a wide variety of semigroups the Arens regularity of $\ell_1(S,\omega)$ implies that S is countable; (see Theorems 3.4 and 3.5).

2. Arens Regularity of $\ell_1(S,\omega)$ and some hereditary properties

Let S be a semigroup and $\omega : S \to (0,\infty)$ be a weight on S, i.e. $\omega(st) \le \omega(s)\omega(t)$ for all $s,t \in S$, and let $\Omega : S\times S \to (0,1]$ be defined by $\Omega(s,t) = \frac{\omega(st)}{\omega(s)\omega(t)}$, for $s,t \in S$. Following

2010 *Mathematics Subject Classification.* 43A10, 43A20, 46H20, 20M18.

Key words and phrases. Arens regularity; weighted semigroup algebra; completely simple semigroup; inverse semigroup.

[3], we call Ω to be $0-$cluster if for each pair of sequences $(x_n),(y_m)$ of distinct elements of S, $\lim_n \lim_m \Omega(x_n,y_m)=0=\lim_m \lim_n \Omega(x_n,y_m)$ whenever both iterated limits exist.

We define,

$$\ell_\infty(S,\omega):=\{f:S\to\mathbb{C}:||f||_{\omega,\infty}=\sup\{|\frac{f(s)}{\omega(s)}|:s\in S\}<\infty\}$$

$$\ell_1(S,\omega):=\{g:S\to\mathbb{C}:||g||_{\omega,1}=\sum_{s\in S}|g(s)|\omega(s)<\infty\}.$$

For ease of reference we quote the following criterion from [3] which will be frequently used in the sequel.

Theorem 2.1. [3, Theorems 3.2, 3.3] *For a weighted semigroup algebra $\ell_1(S,\omega)$, the following statements are equivalent.*

(i) *$\ell_1(S,\omega)$ is regular.*
(ii) *The map $(x,y)\mapsto \chi_A(xy)\Omega(x,y)$ is cluster on $S\times S$ for each $A\subseteq S$.*
(iii) *For each pair of sequences $(x_n),(y_m)$ of distinct points of S there exist subsequences $(x'_n),(y'_m)$ of $(x_n),(y_m)$ respectively such that either*
  (a) *$\lim_n \lim_m \Omega(x'_n,y'_m)=0=\lim_m \lim_n \Omega(x'_n,y'_m)$, or*
  (b) *the matrix $(x'_n y'_m)$ is of type C.*

In particular, if Ω is $0-$cluster then $\ell_1(S,\omega)$ is regular.

Let $\psi:S\to T$ be a homomorphism of semigroups. If ω is a weight on T then trivially $\overleftarrow{\omega}(s):=\omega(\psi(s))$ defines a weight on S.

If $\psi:S\to T$ is an epimorphism and ω is a bounded below (that is, $\inf\omega(S)>0$) weight on S then a direct verification reveals that

$$\overrightarrow{\omega}(t):=\inf\omega(\psi^{-1}(t)),\quad (t\in T),$$

defines a weight on T. We commence with the next elementary result concerning to the stability of regularity under the semigroup homomorphism.

Lemma 2.2. *Let $\psi:S\to T$ be a homomorphism of semigroups.*

(i) *If ψ is onto and ω is bounded below weight on S then the regularity of $\ell_1(S,\omega)$ necessities the regularity of $\ell_1(T,\overrightarrow{\omega})$. Furthermore if Ω is 0-cluster, then $\overrightarrow{\Omega}$ is 0-cluster.*
(ii) *For a weight ω on T if $\ell_1(S,\overleftarrow{\omega})$ is regular, then $\ell_1(T,\omega)$ is regular.*

Proof. (i) Since ω is bounded below, we can assume that, $\inf \omega(S) \geq \varepsilon > 0$, for some $\varepsilon < 1$. Hence $\overrightarrow{\omega} \geq \varepsilon$. Let $(x_n), (y_m)$ be sequences of distinct elements in T. Then there are sequences of distinct elements $(s_n), (t_m)$ in S such that

$$\begin{cases} \overrightarrow{\omega}(x_n) > \omega(s_n)(1-\varepsilon) \quad \text{and} \quad \psi(s_n) = x_n, \\ \overrightarrow{\omega}(y_m) > \omega(t_m)(1-\varepsilon) \quad \text{and} \quad \psi(t_m) = y_m. \end{cases}$$

It follows that $\overrightarrow{\omega}(x_n)\overrightarrow{\omega}(y_m) > \omega(s_n)\omega(t_m)(1-\varepsilon)^2$ and so from $\overrightarrow{\omega}(x_n y_m) \leq \omega(s_n t_m)$ we get $\frac{\overrightarrow{\omega}(x_n y_m)}{\overrightarrow{\omega}(x_n)\overrightarrow{\omega}(y_m)} \leq \frac{1}{(1-\varepsilon)^2}\frac{\omega(s_n t_m)}{\omega(s_n)\omega(t_m)}$; or equivalently,

$$\overrightarrow{\Omega}(x_n, y_m) \leq \frac{1}{(1-\varepsilon)^2}\Omega(s_n, t_m), \quad (n, m \in \mathbb{N}). \tag{2.1}$$

Applying the inequality (2.1), an standard argument based on Theorem 2.1 shows that if $\ell_1(S,\omega)$ is regular then $\ell_1(T, \overrightarrow{\omega})$ is regular. □

Corollary 2.3. *Let $\psi : S \to T$ be a homomorphism of semigroups. If $\ell_1(S)$ is Arens regular then $\ell_1(T,\omega)$ is Arens regular, for every weight function ω on T.*

Proof. Let $\ell^1(S)$ be Arens regular and let ω be a weight on T. Then $\ell^1(S, \overleftarrow{\omega})$ is Arens regular by [3, Corollary 3.4]. Lemma 2.2 implies that $\ell_1(T,\omega)$ is Arens regular. □

3. Arens regularity of $\ell_1(S,\omega)$ and countability of S

We commence with the next result of Craw and Young with a slightly simpler proof.

Corollary 3.1. *(See [4, Corollary 1]) Let S be a countable semigroup. Then there exists a bounded below weight ω on S such that Ω is 0-cluster. In particular, $\ell_1(S,\omega)$ is Arens regular.*

Proof. Let F be the free semigroup generated by the countable semigroup $S = \{a_k : k \in \mathbb{N}\}$. For every element $x \in F$ (with the unique presentation $x = a_{k_1}a_{k_2}\cdots a_{k_r}$) set $\omega_1(x) = 1 + k_1 + k_2 + \cdots k_r$. A direct verification shows that ω_1 is a weight on F with $1 \leq \omega_1$, and that Ω_1 is $0-$cluster. Let $\psi : F \to S$ be the canonical epimorphism. Set $\omega := \overrightarrow{\omega_1}$. By Lemma 2.2, ω is our desired weight on S. □

In the sequel the following elementary lemma will be frequently used.

Lemma 3.2. *A nonempty set X is countable if and only if there exists a function $f : X \to (0,\infty)$ such that the sequence $(f(x_n))$ is unbounded for every sequence (x_n) with distinct elements in X.*

Proof. If $X = \{x_n : n \in \mathbb{N}\}$ is countable the $f(x_n) = n$ is the desired function. For the converse, suppose that such a function $f : X \to (0, \infty)$ exists. Since $X = \cup_{n\in\mathbb{N}}\{x \in X : f(x) \leq n\}$ and each of the sets $\{x \in X : f(x) \leq n\}$ is countable, so X is countable. $\square$

Theorem 3.3. *If $\ell^1(S)$ is not Arens regular and S admits a bounded below weight for which Ω is $0-$cluster, then S is countable.*

Proof. Let ω be a bounded below weight for which Ω is $0-$cluster. Let $\epsilon > 0$ is so that $\omega \geq \epsilon$. Let S be uncountable. By Lemma 3.2 there is a sequence (s_n) of distinct elements in S and $n_0 \in \mathbb{N}$ such that $\omega(s_n) \leq n_0$ for all $n \in \mathbb{N}$. As $\ell_1(S)$ is not Arens regular, there exist subsequences $(s_{n_k}), (s_{m_l})$ of (s_n) such that $\{s_{n_k}s_{m_l} : k < l\} \cap \{s_{n_k}s_{m_l} : k > l\} = \emptyset$ (???????????????). We thus get

$$\Omega(s_{n_k}, s_{m_l}) = \frac{\omega(s_{n_k}s_{m_l})}{\omega(s_{n_k})\omega(s_{m_l})} \geq \frac{\epsilon}{n_0^2}, \quad (k, l \in \mathbb{N}),$$

contradicts the $0-$clusterlity of Ω. $\square$

Abtehi et al. [1] have shown that for a wide variety of semigroups (including Brandt semigroups, weakly cancellative semigroups, (0-)simple inverse semigroups and inverse semigroups with finite set of idempotents) the Arens regularity of the semigroup algebra $\ell^1(S)$ necessities the finiteness of S (see [1, Corollary 3.2, Proposition 3.4 and Theorem 3.6]). Applying these together with Theorem 3.3 we arrive to the next result.

Note that as it has been reminded in Theorem 2.1, if Ω is $0-$cluster then $\ell_1(S, \omega)$ is regular and the converse is also true in the case where S is weakly cancellative; (see [3, Corollary 3.8]).

Theorem 3.4. *If S admits a bounded below weight for which Ω is $0-$cluster then S is countable in either of the following cases.*

(1) *S is a Brandt semigroup.*
(2) *S is weakly cancellative.*
(3) *S is a simple (resp. $0-$simple) inverse semigroup.*
(4) *S is an inverse semigroup with finitely many idempotents.*

In the next result we shall show that the same result holds when S is a completely simple semigroup.

Theorem 3.5. *If S admits a bounded below weight for which Ω is $0-$cluster then S is countable in the case where S is completely simple [resp. 0-simple].*

Proof. Suppose that ω is a bounded below weight on S such that Ω is $0-$cluster. Let S be completely $0-$simple, then as it has been explained in [6], S has the presentation $S \cong M^0(G, I, \Lambda; P) = (I \times G \times \Lambda) \cup \{0\}$, equipped with the multiplication

$$(i, a, \lambda)(j, b, \mu) = \begin{cases} (i, ap_{\lambda j}b, \mu) & \text{if} \quad p_{\lambda j} \neq 0 \\ 0 & \text{if} \quad p_{\lambda j} = 0, \end{cases}$$

$$(i, a, \lambda)0 = 0(i, a, \lambda) = 0.$$

Fix $i_0 \in I$, $\lambda_0 \in \Lambda$ and define $f : I \to (0, \infty)$ by

$$f(i) = \begin{cases} \omega(i, p^{-1}_{\lambda_0 i}, \lambda_0) & \text{if} \quad p_{\lambda_0 i} \neq 0 \\ \omega(i, 1, \lambda_0) & \text{if} \quad p_{\lambda_0 i} = 0. \end{cases}$$

Let (i_n) be a sequence of distinct elements in I and set

$$x_n = \begin{cases} (i_n, p^{-1}_{\lambda_0 i_n}, \lambda_0) & \text{if} \quad p_{\lambda_0 i_n} \neq 0 \\ (i_n, 1, \lambda_0) & \text{if} \quad p_{\lambda_0 i_n} = 0. \end{cases}$$

It is readily verified that if $p_{\lambda_0 i_n} \neq 0$ then $x_n x_m = x_n$, for all $m \in \mathbb{N}$; indeed

$$x_n x_m = (i_n, p^{-1}_{\lambda_0 i_n}, \lambda_0)(i_m, p^{-1}_{\lambda_0 i_m}, \lambda_0) = (i_n, p^{-1}_{\lambda_0 i_n} p_{\lambda_0 i_m} p^{-1}_{\lambda_0 i_m}, \lambda_0) = (i_n p^{-1}_{\lambda_0 i_n}, \lambda_0) = x_n.$$

And if $p_{\lambda_0 i_n} = 0$ then $x_n x_m = 0$, for all $m \in \mathbb{N}$.

Hence $\frac{1}{f(i_m)} = \frac{1}{\omega(x_m)} = \frac{\omega(x_n x_m)}{\omega(x_n)\omega(x_m)} = \Omega(x_n, x_m)$ in the case where $p_{\lambda_0 i_n} \neq 0$ and $(\frac{\omega(0)}{f(i_m)})^2 = (\frac{\omega(0)}{\omega(x_m)})^2 = \frac{\omega(0)}{\omega(x_n)\omega(x_m)}$ whenever $p_{\lambda_0 i_n} = 0$. These observations together with the $0-$clusterlity of Ω imply that $(f(i_m))$ is unbounded. Hence I is countable, by Lemma 3.2. Similarly Λ is countable. We are going to show that G is also countable. To this end, let $\omega_0(g) = \omega(i_0, gp^{-1}_{\lambda_0, i}, \lambda_0) \quad (g \in G)$. Then ω_0 is a weight on G such that Ω_0 is $0-$cluster and so G is countable, by Theorem 3.4. Therefore S is countable as claimed. Proof for the case that S completely simple semigroup is similar. $\square$

Acknowledgments. This research was supported by the Centers of Excellence for Mathematics at the University of Isfahan.

References

[1] F. Abtahi, B. Khodsiani and A. Rejali, *The Arens regularity of inverse semigroup algebras,* Preprint. 3

[2] R. Arens, *The adjoint of a bilinear operation,* Proc. Amer. Math. Soc. **2**,(1951), 839-848. 1

[3] J.W. Baker and A. Rejali, *On the Arens regularity of weighted convolution algebras,* J. London Math. Soc. (2) **40**(1989), 535-546. 1, 2, 2.1, 2, 3

[4] I.G. Craw and N.J. Young, *Regularity of multiplications in weighted group and semigroup algebras,* Quart. J. Math. **25** (1974), 351-358. (document), 1, 3.1

[5] H.G. Dales and A.T.M. Lau, *The Second Duals of Beurling Algebras,* Memoris American Math. Soc., **177**, 1-191, (2005). 1

[6] J.M. Howie, *An Introduction to Semigroup Theory,* Academic Press, (1976). 1, 3

[7] N.J. Young, *Semigroup algebras having regular multiplication,* Stadia Math. **47**(1973), 191-196. 1

[1,2] Department of Mathematics, University of Isfahan, Isfahan, IRAN.

E-mail address: `b-khodsiani@sci.ui.ac.ir`

E-mail address: `rejali@sci.ui.ac.ir`

[3] Department of Pure Mathematics and Centre of Excellence in Analysis on Algebraic Structures (CEAAS), Ferdowsi University of Mashhad, P.O. Box 1159, Mashhad 91775, Iran.

E-mail address: `vishki@um.ac.ir`